Complex Variable Boundary Element Method for External Potential Flows

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Abstract

THE paper describes a simple, efficient complex variable boundary element method algorithm for calculating potential flow past single and multicomponent airfoils in free air, in ground effect, in an infinite cascade, and in solid, open, and perforated wall wind tunnels. The theoretical development is based on the representation of the complex disturbance velocity by the Cauchy-type integral, formulating the airfoil problem as an exterior (Riemann-)Hilbert problem and accounting for the outer constraints using the concept of Green's function in the complex plane.

Contents

The Complex Variable Boundary Element Method (CVBEM), based on the application of the Cauchy-type integral to two-dimensional potential problems, has emerged recently as a new discipline in computational fluid dynamics and hydrology.[1-3]

The prototype problem is that of determining an analytic function (complex potential) inside a domain from the values of either the real part (potential) or the imaginary part (stream function) prescribed on the boundary. This specifies an interior Schwarz problem, consisting in finding a solution of the interior Dirichlet problem for the quantity prescribed and integration of the Cauchy-Riemann equations to obtain the other (conjugate) quantity. The Schwarz problem is thus solved to within an arbitrary constant. The solution is unique if the other quantity is defined at one point of the domain or boundary. Typical applications of the CVBEM involve mixed boundary value problems for internal flows, where the boundary is a combination of streamlines and equipotential lines.

The extension of the method to external potential flows, given here, is based on representing the complex disturbance velocity by the Cauchy-type integral

$$w(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta \tag{1}$$

where C is the counterclockwise oriented, simple closed airfoil contour (Fig. 1), and f is a complex-valued density function. Correspondence between the Cauchy-type integral, Eq. (1), and the distribution of sources and vortices

$$w(z) = \int_C \left[\frac{\sigma(\zeta)}{2\pi(z-\zeta)} + \frac{i\gamma(\zeta)}{2\pi(z-\zeta)} \right] |d\zeta| \tag{2}$$

is provided by the relationship

$$f(\zeta) = -\left[\sigma(\zeta) + i\gamma(\zeta)\right] e^{-i\nu(\zeta)} \tag{3}$$

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The real-valued functions σ and γ are the source and the vortex densities, respectively, and ν is the angle between the exterior normal to C and the real axis.

There are, obviously, a multiplicity of density functions f capable of representing a given analytic function w in either the internal domain D^+ or the external domain D^-. For the external flow problem, a natural choice is to consider $f(\zeta)$ as a boundary value of the function $f(z)$ analytic in D^- and continuous in $D^- \cup C$. Then, by the Cauchy integral formula,

$$w(z) = \begin{cases} f(\infty) - f(z), & z \in D^- \cup C \\ f(\infty), & z \in D^+ \end{cases}$$

where $f(\infty)$ is the value of $f(z)$ as $|z| \to \infty$.

If the freestream velocity is of unit magnitude and angle α with respect to the real axis, the (total) complex velocity will be

$$W(z) = e^{-i\alpha} + w(z) \tag{4}$$

Choosing $f(\infty) + e^{-i\alpha} = 0$, the fictitious interior flow vanishes, and on the exterior side of the contour

$$W(\zeta) = -f(\zeta) \tag{5}$$

Since

$$W(\zeta) = \left[V_n(\zeta) - iV_t(\zeta)\right] e^{-i\nu(\zeta)} \tag{6}$$

Eq. (5) also implies that

$$V_n(\zeta) = \sigma(\zeta), \qquad V_t(\zeta) = -\gamma(\zeta) \tag{7}$$

Thus, for zero interior flow, the external normal and tangential velocities are equal to the source and (minus) vortex densities, respectively. (The opposite is also true.)

Combining Eqs. (4–7), the following airfoil boundary condition is obtained

$$\mathrm{Re}\left\{\left[e^{-i\alpha} + w(\zeta)\right] e^{i\nu(\zeta)}\right\} = \sigma(\zeta)$$

where the right side is the prescribed normal (transpiration) velocity. The alternative

$$\mathrm{Re}\left[\frac{w(\zeta)}{q(\zeta)}\right] = c(\zeta) \tag{8}$$

where

$$q(\zeta) = e^{-i\nu(\zeta)}, \qquad c(\zeta) = \sigma(\zeta) - \cos\left[\nu(\zeta) - \alpha\right]$$

are prescribed functions on the contour C, is identified as that specifying the Hilbert[4] or Riemann-Hilbert[5] problem for the function $w(z)$ analytic in D^-. It resembles the Schwarz problem, except that the term in the square brackets of Eq. (8) is not a boundary value of a function analytic in D^-. The proof of existence and (non)uniqueness can be obtained using Gakhov's method[4] of regularizing factor. The solution exists in view of the fact that the index

$$\kappa = \frac{1}{2\pi} \arg\left[q(\zeta)\right]_C = \frac{1}{2\pi}\left[-\nu(\zeta)\right]_C = -1$$

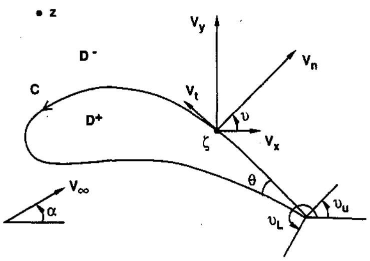

Fig. 1 Interior and exterior domains.

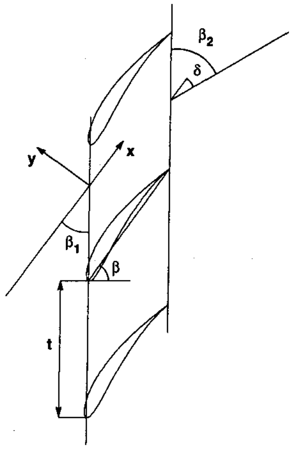

Fig. 2 Geometry of the Gostelow cascade.

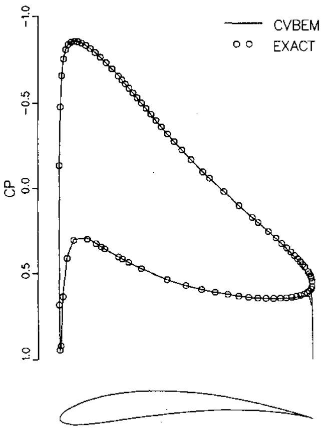

Fig. 3 Gostelow blade: comparison of exact and CVBEM solutions.

is negative and can be constructed explicitly as a linear combination depending on $-2\kappa+1=3$ arbitrary constants. Two of the constants are complex conjugates and vanish by the virtue of the far-field condition $w(\infty)=0$. The solution is made unique by selecting the third constant, which, as may have been expected, is equivalent to specifying the circulation around a smooth contour or to satisfying the Kutta-Joukowski condition for an airfoil having a trailing edge.

Since the Cauchy-type integral, Eq. (1), is nonsingular at contour points where f is continuous, the Kutta-Joukowski condition is synonymous with the requirement of equal values of Cauchy density at the upper and lower sides of the trailing edge, $f_U=f_L$. From Eq. (3), in terms of the source and vortex densities,

$$\sigma_L + i\gamma_L = -(\sigma_U + i\gamma_U)e^{i\theta} \tag{9}$$

where $\theta = \nu_L - \nu_U - \pi$ is the trailing-edge angle.

Internal flow problems, described by linear homogeneous or periodic boundary conditions at the outer boundary, are treated as external flow problems replacing Eq. (2) by

$$w(z) = \int_C \left[\sigma(\zeta)G_\sigma(z,\zeta) + \gamma(\zeta)G_\gamma(z,\zeta)\right] |\mathrm{d}\zeta|$$

where C is the inner (airfoil) contour and

$$G_\sigma(z,\zeta) = \frac{1}{2\pi(z-\zeta)} + H_\sigma(z,\zeta)$$

$$G_\gamma(z,\zeta) = \frac{i}{2\pi(z-\zeta)} + H_\gamma(z,\zeta)$$

are the Green's functions, satisfying the outer boundary conditions. The analytic parts H_σ and H_γ associated with the constraints imposed by wind tunnels with solid, porous, and open jet boundaries and an infinite cascade are given in the full paper.

The CVBEM algorithm, which approximates Eq. (1) by a sum of individual contribution of straight-line boundary elements, is developed using linear trial functions with nodal points at element endpoints and collocation points at element midpoints. The densities σ and γ are continuous at the element endpoints, except at the trailing edge, where f has to be continuous. Since Eq. (9) represents two real-valued equations, the resultant system of linear equations is overdetermined by 1 per each airfoil trailing edge and has to be solved as such. The tangential velocities at the nodal points are obtained from Eqs. (7) as the local values of vortex density.

The algorithm is simple, efficient, and like those of other boundary element methods, easily adaptible to microcomputers. No error analysis is offered; however, the accuracy of the method is demonstrated amply on examples where the conformal-mapping solutions are known. One of them is the Gostelow compressor cascade[6] with the following geometrical parameters: spacing to chord ratio $t/c=0.99$, stagger angle $\beta=52.5$ deg, and inlet angle $\beta_1=36.5$ deg (see Fig. 2). The agreement of the CVBEM pressure distribution (88 nodes) with the theoretical one is demonstrated in Fig. 3; also the calculated outlet angle $\beta_2=59.95$ deg compares well with Gostelow's value of 59.98 deg.

Comparisons like this indicate that the method is capable of providing almost exact solutions, which can then be used as incompressible-flow test cases or initial flowfield values for compressible-flow computations.

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